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Abstract. We derive the critical behaviour of a cellular automaton traffic flow model using an order parameter breaking the symmetry of the jam-free phase. Random braking appears to be the symmetry-breaking field conjugate to the order parameter. For $v_{\max} = 2$, we determine the values of the critical exponents β, γ and δ using an order-3 cluster approximation and computer simulations. These critical exponents satisfy a scaling relation, which can be derived assuming that the order parameter is a generalized homogeneous function of $|\rho - \rho_c|$ and p in the vicinity of the phase transition point.

1. Introduction

Cellular automation (CA) models of traffic flow have attracted much interest since the publication of the Nagel–Schreckenberg (NS) model [1]. The NS model is a probabilistic CA model of traffic flow on a one-lane highway. The road is represented by a lattice of L sites with periodic boundary conditions. Each site is either empty (in state e) or occupied by a car with velocity $v = 0, 1, 2, \ldots, v_{\max}$ (in state v). If d_i is the distance between car i and car $i + 1$ (cars are moving to the right), velocities are updated in parallel according to the following two subrules:

$$v_i(t + \tfrac{1}{2}) = \min(v_i(t) + 1, d_i(t) - 1, v_{\max}) \tag{1}$$

$$v_i(t + 1) = \max(v_i(t + 1/2) - 1, 0) \qquad \text{with probability } p \tag{2}$$

where $v_i(t)$ is the velocity of car i at time t; then, cars are moving according to the subrule

$$x_i(t + 1) = x_i(t) + v_i(t + 1) \tag{3}$$

where $x_i(t)$ is the position of car i at time t. The model contains three parameters: the maximum speed $v_{\max}$, which is the same for all cars, the braking probability p, and the car density ρ. For a clear presentation of the various approximate techniques to calculate relevant physical quantities of the NS model compared with results obtained from computer simulations see [2].

If $p = 0$, the NS model is deterministic and the average velocity over the whole lattice is exactly given by

$$\langle v \rangle = \min\left(v_{\max}, \frac{1}{\rho} - 1 \right). \tag{4}$$

This expression shows that, below a critical car density $\rho_c = 1/(v_{max} + 1)$, all cars move with a velocity equal to v_{max}, while above ρ_c, the average velocity is less than v_{max}. This transition from a free-moving regime to a congested regime is usually viewed as a second-order phase transition.

If $p \neq 0$, except in the case $v_{max} = 1$, no exact expression for the average velocity has been obtained, and there is no consensus concerning the existence of the phase transition (see, for instance, [3] and [4] and references therein). One point that is particularly unclear is the definition of the order parameter.

If CA models of traffic flow exhibit second-order phase transitions, we have to understand the nature of the order parameter, show how it is related to symmetry breaking, determine the symmetry-breaking field conjugate to the order parameter, define the analogue of the susceptibility, study the critical behaviour, and, possibly, find scaling laws.

The purpose of this short note is to fulfil this program. In our discussion we shall consider a slightly simpler model of traffic flow. We shall assume that the acceleration, which is equal to 1 in the NS model, has the largest possible value (less or equal to v_{max}) as in the Fukui–Ishibashi (FI) model [5], except that these authors apply random delays only to cars whose velocity is equal to v_{max}. That is, in our model, we just replace (1) by

$$v_i(t + \tfrac{1}{2}) = \min(d_i(t) - 1, v_{max}). \tag{5}$$

2. Symmetry considerations

Deterministic CA rules modelling traffic flow on one-lane highways are number-conserving ($\rho = $ constant). Limit sets of number-conserving CA have, in most cases, a very simple structure and, these limit sets are reached after a number of time steps proportional to the lattice size [6–8]. The limit set of our model is identical to the limit set of the NS or FI models.

If $\rho < \rho_c$, any configuration in the limit set consists of 'perfect tiles' of $v_{max} + 1$ cells as shown below:

| e | e | $\cdots$ | e | v_{max} |
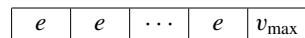
in a sea of cells in state e. In particular, it can be shown [9] that the probability to find in the limit set a sequence of $v_{max} + 1$ empty sites is given exactly by
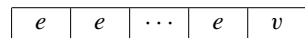
$$P(\underbrace{ee \cdots e}_{v_{max}}) = 1 - (v_{max} + 1)\rho.$$

This relation is a simple consequence of the following argument: if, in an empty road, you add $L\rho$ cars, in the limit set, for $\rho < \rho_c$, we have

$$LP(\underbrace{ee \cdots e}_{v_{max}}) = L - (v_{max} + 1)L\rho$$

where L is the lattice size.

If $\rho > \rho_c$, a configuration belonging to the limit set only consists of a mixture of tiles containing $v + 1$ cells of the type

| e | e | $\cdots$ | e | v |

where $v = 0, 1, \ldots, v_{max}$. If $\{\rho_v \mid v = 0, 1, 2, \ldots, v_{max}\}$ is the velocities distribution, we have
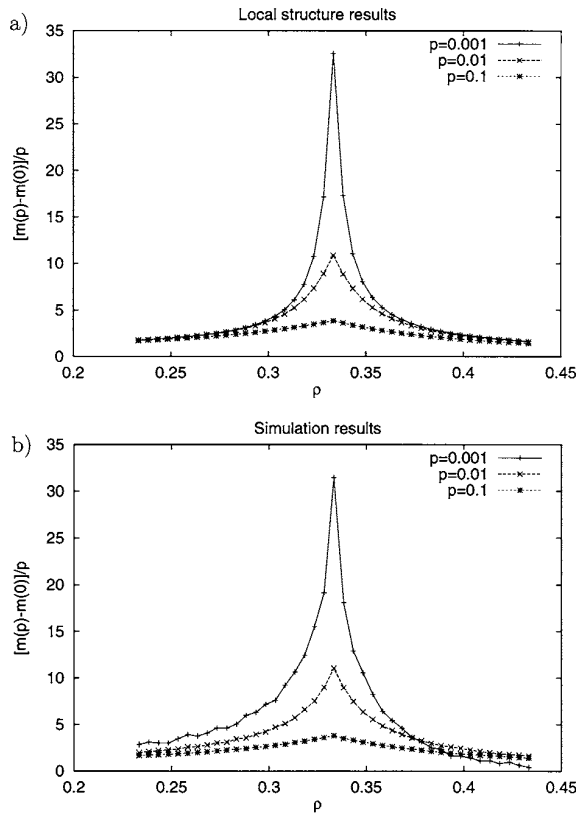
$$\rho = \sum_{v=0}^{v_{max}} \rho_v$$

Figure 1. Divergence of the susceptibility at the critical point using local structure approximation (*a*) and data obtained from simulations (*b*). Excellent agreement of simulations and local structure approximation is clearly visible.

$$1 = \sum_{v=0}^{v_{\max}} (v+1)\rho_v$$

$$\langle v \rangle = \frac{1}{\rho}\left(\sum_{v=0}^{v_{\max}} v\rho_v\right).$$

Note that (4) is a simple consequence of these relations.

If we introduce random braking, then, even at low density, some tiles become defective, which causes the average velocity to be less than $v_{\max}$. The random-braking parameter p can, therefore, be viewed as a symmetry-breaking field, and the order parameter, conjugate to that field is

$$m = v_{\max} - \langle v \rangle. \tag{6}$$

This point of view implies that the phase transition characterized by m will be smeared out in the presence of random braking as in ferromagnetic systems placed in a magnetic field.

From (6) and (4), it follows that, for $p = 0$,

$$m = \begin{cases} 0 & \text{if} \quad \rho \leqslant \rho_c \\ \dfrac{\rho - \rho_c}{\rho\rho_c} & \text{otherwise.} \end{cases} \tag{7}$$

Hence, the critical exponent β is equal to 1.

a)

Local structure results
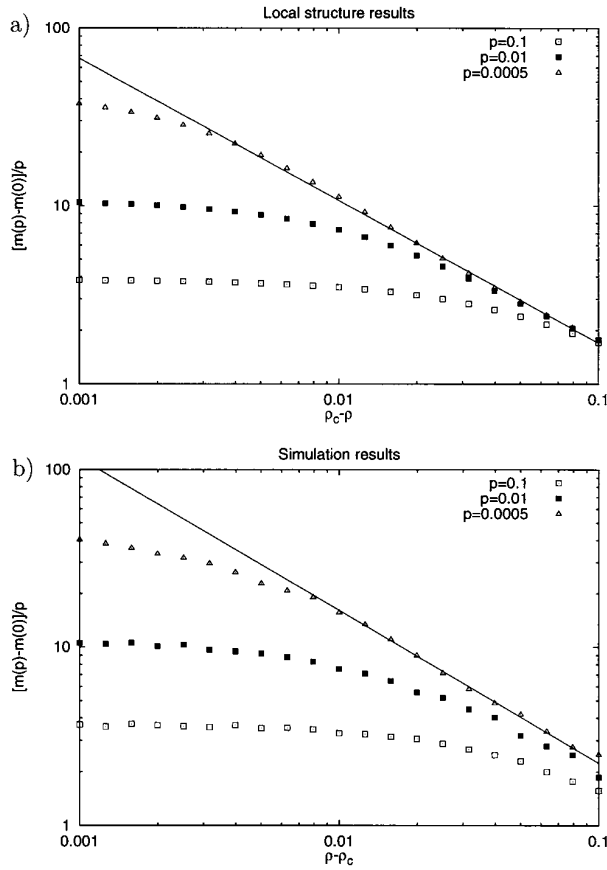

b)

Simulation results

Figure 2. Plot of $[m(p) - m(0)]/p$ as a function of $|\rho - \rho_c|$ for $\rho < \rho_c$ using local structure approximation (*a*) and data obtained from simulations (*b*). The straight line represents the best fit for $p = 0.0005$ and $0.01 < \rho_c - \rho < 0.1$. We obtained γ by performing such a fit for several different values of p and extrapolation to $p = 0$.

3. Approximate techniques

To determine the other critical exponents, we have used local structure approximations as described in [10] and numerical simulations.

To construct a local structure approximation, it is more convenient to represent configurations of cars as binary sequences, where zeros represent empty spaces and ones represent cars. Since for $v_{max} = 2$ the speed of a car is determined by the states of, at most, two sites in front of it, the minimal block size to obtain nontrivial results is three (the site occupied by a car plus two sites in front of it). In what follows, we limit our attention to order-3 local structure approximation.

Using three-block probabilities, we can write a set of equations describing the time evolution of these probabilities

$$P_{t+1}(b_2 b_3 b_4) = \sum_{\substack{a_i \in \{0,1\} \\ i=0,1,\dots,6}} w(b_2 b_3 b_4 | a_0 a_1 a_2 a_3 a_4 a_5 a_6) P_t(a_0 a_1 a_2 a_3 a_4 a_5 a_6) \qquad (8)$$

where $P_t(b_2 b_3 b_4)$ is the probability of block $b_2 b_3 b_4$ at time t, and $w(b_2 b_3 b_4 | a_0 a_1 a_2 a_3 a_4 a_5 a_6)$ is the conditional probability that the rule maps the seven-block $a_0 a_1 a_2 a_3 a_4 a_5 a_6$ into the three-block $b_2 b_3 b_4$. Letters a denote the states of lattice sites at time t, while b denote states at time $t + 1$, so that, for example, a_3 is the state of site $i = 3$ at time t, and b_3 is the state of the same
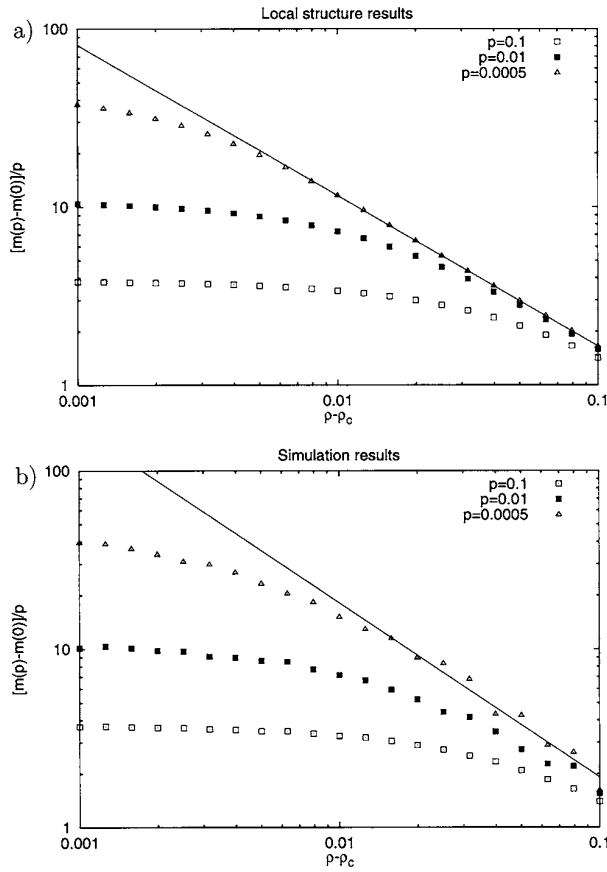
Figure 3. Plot of $[m(p)-m(0)]/p$ as a function of $|\rho - \rho_c|$ for $\rho > \rho_c$ using local structure approximation (*a*) and data obtained from simulations (*b*). The straight line represents the best fit for $p = 0.0005$ and $0.01 < \rho - \rho_c < 0.1$. We obtained γ' by performing such a fit for several different values of p and extrapolation to $p = 0$.

site at time $t + 1$. Conditional probabilities w can be easily computed from the definition of the rule, although, since their number is quite large, we used a computer program for their determination.

Equation (8) is exact. The approximation consists of expressing the seven-block probabilities in terms of three-block probabilities using Bayesian extension. That is,

$$P_t(a_0a_1a_2a_3a_4a_5a_6) = \frac{P_t(a_0a_1a_2)\,P_t(a_1a_2a_3)\,P_t(a_2a_3a_4)\,P_t(a_3a_4a_5)\,P_t(a_4a_5a_6)}{P_t(a_1a_2)\,P_t(a_2a_3)\,P_t(a_3a_4)\,P_t(a_4a_5)} \tag{9}$$

where $P_t(a_ia_{i+1}) = P_t(a_ia_{i+1}0) + P_t(a_ia_{i+1}1)$ for $i = 1, \ldots, 4$. Equations (8) and (9) define a dynamical system whose fixed point approximates three-block probabilities of the limit set of the CA rule. Due to the nonlinear nature of these equations it is not possible to find the fixed point analytically. We have determined it numerically.

From the knowledge of three-block probabilities $P_t(a_2a_3a_4)$, the car density is given by

$$\rho = P_t(1) = P_t(100) + P_t(101) + P_t(110) + P_t(111) \tag{10}$$

and the average velocity by

$$\langle v \rangle = \frac{2(1 - p)P_t(100) + pP_t(100) + (1 - p)P_t(101)}{\rho}. \tag{11}$$

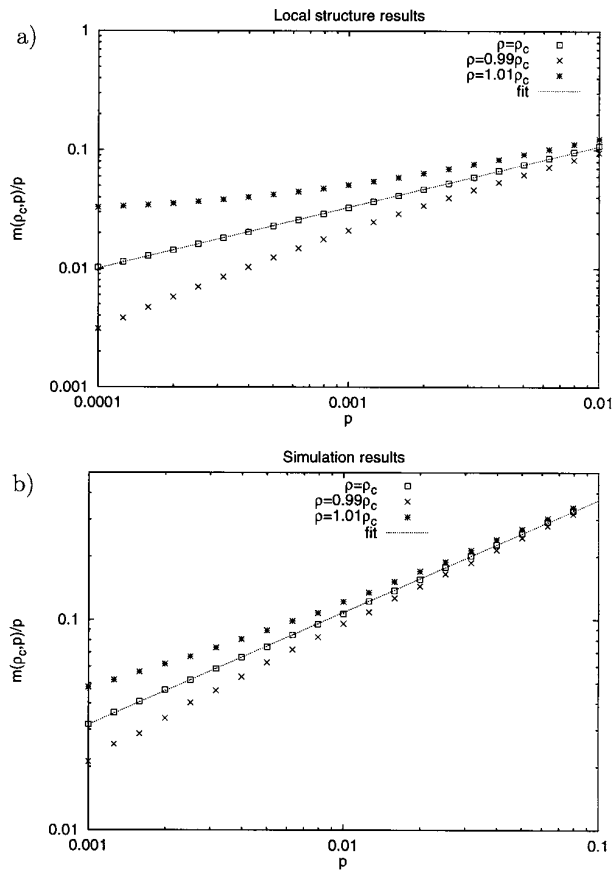This last result gives the expression of the order parameter $m = v_{\max} - \langle v \rangle$.

Figure 4. Order parameter m at the critical point $\rho = \rho_c$ as a function of p using local structure approximation (*a*) and data obtained from simulations (*b*). The slope of the best fit line at $\rho = \rho_c$ yields $1/\delta$.

Note that $\rho = P_t(1)$ is a constant of motion for equation (8) whose fixed point depends, therefore, upon the initial car density $P_0(1) = P_0(100) + P_0(101) + P_0(110) + P_0(111)$. Consequently, using order-3 local structure approximation allows one to determine the order parameter m as a function of car density, and the critical exponents.

For most simulations, we used a lattice size equal to 1000 and our results are average of 1000 runs of 1000 iterations. For $p = 0.0005$ we took a lattice of 10 000 sites and averaged 500 runs of 10 000 iterations.

4. Results

The susceptibility χ_ρ at constant ρ, defined by

$$\chi_\rho = \lim_{p \to 0} \frac{\partial m}{\partial p} \tag{12}$$

cannot be calculated exactly. So we determined its value using local structure approximation as described in the preceding section and computer simulations for $v_{max} = 2$. Figures 1(*a*) and (*b*) show the results obtained from order-3 local structure approximation† and simulations,

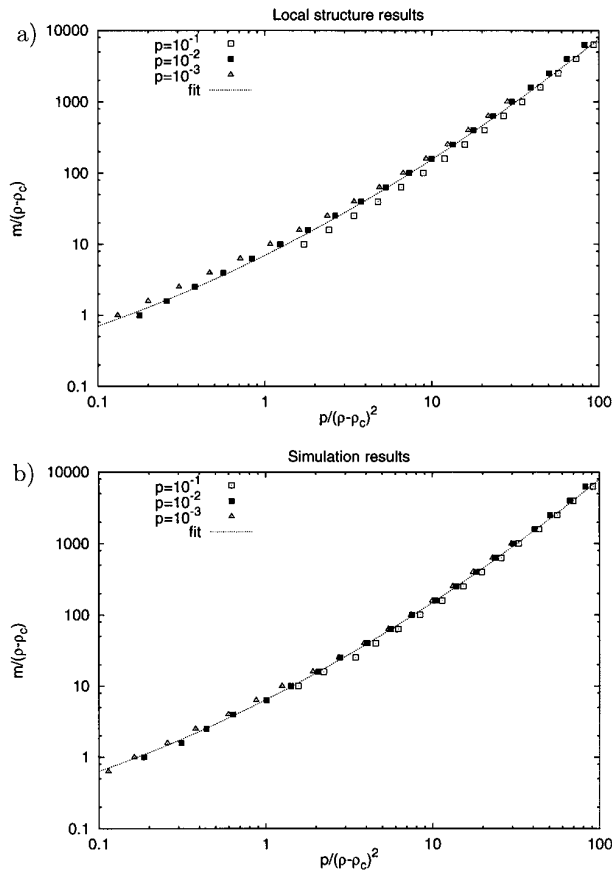† If $v_{max} > 2$, one should use $(v_{max} + 1)$th-order local structure approximation.

Figure 5. Illustration of the existence of universal scaling function for $\rho < \rho_c$ using local structure approximation (*a*) and data obtained from simulations (*b*).

respectively. In the limit $p \to 0$, the susceptibility diverges as $(\rho_c - \rho)^{-\gamma}$ for $\rho < \rho_c$, and as $(\rho - \rho_c)^{-\gamma'}$ for $\rho > \rho_c$. Using local structure approximation (figures 2(*a*) and 3(*a*)), we found

$$\gamma = 0.91 \pm 0.03 \qquad \text{and} \qquad \gamma' = 0.98 \pm 0.03$$

while simulations (figures 2(*b*) and 3(*b*)) yield

$$\gamma = 0.86 \pm 0.05 \qquad \text{and} \qquad \gamma' = 0.94 \pm 0.05.$$

Another exponent of interest is δ. It characterizes the behaviour of m as a power of p for $\rho = \rho_c$. Here again we have determined the value of

$$\lim_{p \to 0} \frac{m(\rho_c, 0) - m(\rho_c, p)}{p}$$

using order-3 local structure approximation and simulations. Our results are represented in figure 4, and the values of δ, obtained using order-3 local structure approximation and computer simulations, are, respectively, given by

$$\frac{1}{\delta} = 0.51 \pm 0.01 \qquad \text{and} \qquad \frac{1}{\delta} = 0.53 \pm 0.02$$

It is interesting to note that the values

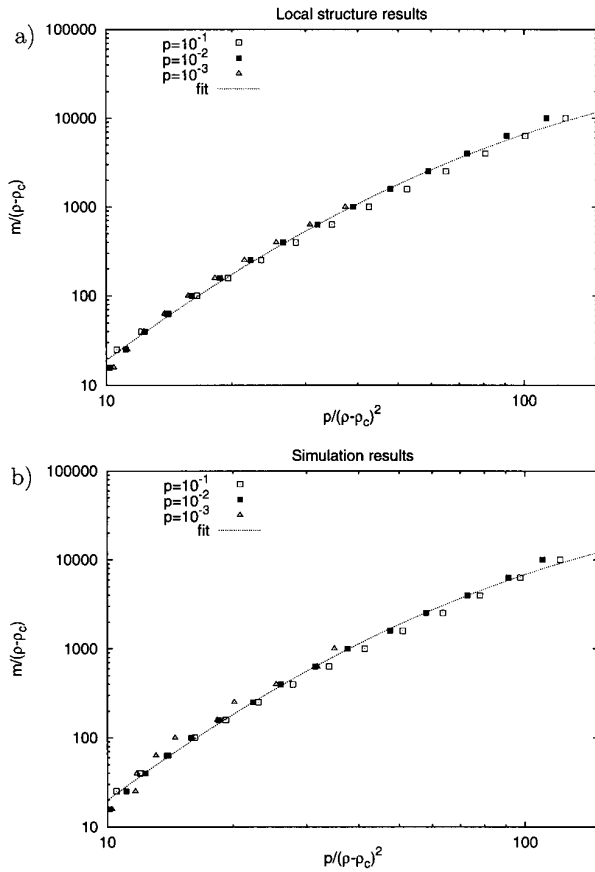$$\beta = 1 \qquad \gamma \approx 1 \qquad \delta \approx 2$$

Figure 6. Illustration of the existence of universal scaling function for $\rho > \rho_c$ using local structure approximation (*a*) and data obtained from simulations (*b*).

obtained for the critical exponents are found in equilibrium statistical physics in the case of second-order phase transitions characterized by non-negative order parameters above the upper critical dimensionality.

Close to the phase transition point, critical exponents obey scaling relations. If we assume that, in the vicinity of the critical point ($\rho = \rho_c$, $p = 0$), the order parameter m is a generalized homogeneous function of $\rho - \rho_c$ and p of the form

$$m = |\rho - \rho_c|^\beta f \left(\frac{p}{|\rho - \rho_c|^{\beta\delta}} \right) \tag{13}$$

where the function f is such that $f(0) \neq 0$, then, differentiating f with respect to p and taking the limit $p \to 0$, we readily obtain

$$\gamma = \gamma' = (\delta - 1)\beta \tag{14}$$

which is verified by our numerical values obtained either using order-3 local structure approximation or computer simulations. Figures 5 and 6, clearly confirm the existence of a universal scaling function.

5. Conclusion

In this short paper, we have derived the critical behaviour of a CA traffic flow model using an order parameter breaking the symmetry of the jam-free phase. Random braking, which is thought to be an essential ingredient of any CA traffic flow model, appears to be the symmetry-breaking field conjugate to the order parameter. For $v_{max} = 2$, we have determined the critical exponents β, γ and δ using order-3 local structure approximation and computer simulations. These critical exponents satisfy a scaling relation which can be derived assuming that the order parameter is a generalized homogeneous function of $|\rho - \rho_c|$ and p in the vicinity of the phase transition point.

Acknowledgments

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